

Some results on a conjecture of Voisin for surfaces of geometric genus one

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Abstract Inspired by the Bloch–Beilinson conjectures, Voisin has formulated a conjecture concerning the Chow group of 0–cycles on complex varieties of geometric genus one. This note presents some new examples of surfaces for which Voisin’s conjecture is verified.

Keywords Algebraic cycles · Chow groups · motives · finite–dimensional motives · $K3$ surfaces · surfaces of general type

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1 Introduction

The world of algebraic cycles on complex varieties is famous for its open questions (fairly comprehensive tourist guides, nicely exhibiting the boundaries between what is known and what is not known, can be found in [53] and [34]). The Bloch–Beilinson conjectures predict that this world has beautiful structure, and more precisely that there exists an intimate relation between Chow groups (i.e., algebraic cycles modulo rational equivalence) and singular cohomology.

The present note focuses on one particular instance of this predictive power of the Bloch–Beilinson conjectures: we consider the case of algebraic cycles on self–products $X \times X$, where X is an n –dimensional smooth complex projective variety with $h^{n,0} = 1$ and $h^{i,0} = 0$ for all $0 < i < n$. The Chow group of 0–cycles

$$A^{2n}(X \times X)$$

is conjecturally related to the cohomology groups

$$H^{4n}(X \times X), H^{4n-1}(X \times X), \dots, H^{2n}(X \times X).$$

Let

$$\iota: X \times X \rightarrow X \times X$$

denote the involution exchanging the two factors. Then a consequence of this conjectural relation is that the effect of ι on $A^{2n}(X \times X)$ should be a reflection of the effect of ι on

$$H^{4n}(X \times X), H^{4n-1}(X \times X), \dots, H^{2n}(X \times X).$$

Now, the condition $h^{n,0}(X) = 1$ ensures that the action of ι on $H^{2n}(X \times X)$ is particularly well-understood: we have that

$$(\text{id} + (-1)^{n+1}\iota_*)H^{2n}(X \times X) \subset H^{2n}(X \times X) \cap F^1,$$

where F^* denotes the Hodge filtration (cf. lemma 6 below). Conjecturally, this implies that

$$\text{id} = (-1)^n \iota_*: \text{Gr}_F^{2n} A^{2n}(X \times X) \rightarrow \text{Gr}_F^{2n} A^{2n}(X \times X),$$

where Gr_F^{2n} denotes the deepest level of the conjectural Bloch–Beilinson filtration on Chow groups. The condition on the Hodge numbers $h^{i,0}$ implies that all the levels Gr_F^j for $j < 2n$ are conjecturally 0. Thus, one arrives at the following explicit conjecture concerning 0-cycles on $X \times X$, which was first formulated by Voisin:

Conjecture 1 (Voisin [47]) Let X be a smooth projective complex variety of dimension n , with $h^{n,0}(X) = 1$ and $h^{j,0}(X) = 0$ for $0 < j < n$. Let $z, z' \in A^n X$ be 0-cycles of degree 0. Then

$$z \times z' = (-1)^n z' \times z \text{ in } A^{2n}(X \times X).$$

(The notation $z \times z'$ is a short-hand for the cycle class $(p_1)^*(z) \cdot (p_2)^*(z') \in A^{2n}(X \times X)$, where p_1, p_2 denote projection on the first, resp. second factor.)

Loosely speaking: we have that almost all 0-cycles are $(-1)^n \iota$ -invariant. Conjecture 1 is proven by Voisin for Kummer surfaces, and for a certain 10-dimensional family of $K3$ surfaces [47], obtained by desingularizing a double cover of \mathbb{P}^2 branched along 2 cubics.

The aim of this note is to add some more cases to the list of examples where conjecture 1 is verified. The main ingredient we use is the theory of finite-dimensional motives of Kimura and O’Sullivan [28], [1], which did not exist at the time [47] was written.¹

Proposition (=propositions 5, 27, 21, 14 and 16)) *Let X be one of the following:*

- (i) *a surface with $p_g = 1$, $q = 0$ which is ρ -maximal (in the sense of [3]) and has finite-dimensional motive (in the sense of [28]);*
- (ii) *a Kunev surface [41];*
- (iii) *a $K3$ surface with a Shioda–Inose structure (for example, a $K3$ with Picard number 19 or 20);*
- (iv) *a $K3$ surface obtained from a double cover of \mathbb{P}^2 branched along the union of an irreducible quadric and an irreducible quartic;*
- (v) *a $K3$ surface obtained from a double cover of \mathbb{P}^2 branched along 6 lines.*

Then conjecture 1 is true for X .

Some explicit examples of families of surfaces of general type satisfying hypothesis (i) are given in remark 11. A Kunev surface is a certain surface of general type with $q = 0$ and $p_g = 1$, these surfaces form a 12-dimensional family [41] (cf. definition 27 for a precise definition). The generic member of a $K3$ surface as in (iv) has Picard number 9. I am not aware of any $K3$ surface of Picard number less than 9 for which conjecture 1 is known, so obviously there is a lot of work remaining to be done !

Conventions *In this note, all varieties will be quasi-projective irreducible algebraic variety over \mathbb{C} , endowed with the Zariski topology. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients: *we will denote by $A_j(X)$ the Chow group of j -dimensional cycles on X with \mathbb{Q} -coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ will be used interchangeably.*

The notation $A_{hom}^j(X)$, resp. $A_{AJ}^j(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f: X \rightarrow Y$, we will write $\Gamma_f \in A_(X \times Y)$ for the graph of f .*

In an effort to lighten notation, we will often write $H^j(X)$ or $H_j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$ resp. Borel–Moore homology $H_j(X, \mathbb{Q})$.

¹ Though reading with hindsight, it is clear that [47] already contains, *avant la lettre*, many of the ideas of the theory of finite-dimensional motives – in particular, the idea of considering the action of the symmetric group S_k on the Chow groups of the product X^k .

2 Finite-dimensional motives

We refer to [28], [1], [21], [34] for the definition of finite-dimensional motive. What mainly concerns us here is the nilpotence theorem, which embodies a crucial property of varieties with finite-dimensional motive:

Theorem 2 (Kimura [28]) *Let X be a smooth projective variety of dimension n with finite-dimensional motive. Let $\Gamma \in A^n(X \times X)$ be a correspondence which is numerically trivial. Then there is $N \in \mathbb{N}$ such that*

$$\Gamma^{\circ N} = 0 \quad \in A^n(X \times X) .$$

Actually, the nilpotence property (for powers of X) could serve as an alternative definition of finite-dimensional motive, as shown by a result of Jannsen [26, Corollary 3.9].

Conjecturally, any variety has finite-dimensional motive [28]. We are still far from knowing this, but at least there are quite a few non-trivial examples:

Remark 3 The following varieties have finite-dimensional motive: varieties dominated by products of curves [28], K3 surfaces with Picard number 19 or 20 [36], surfaces not of general type with vanishing geometric genus [20, Theorem 2.11], Godeaux surfaces [20], Catanese and Barlow surfaces [52], many examples of surfaces of general type with $p_g = 0$ [37], Hilbert schemes of surfaces known to have finite-dimensional motive [11], generalized Kummer varieties [54, Remark 2.9(ii)], 3-folds with nef tangent bundle [22] or [44, Example 3.16], 4-folds with nef tangent bundle [23], log-homogeneous varieties in the sense of [8] (this follows from [23, Theorem 4.4]), certain 3-folds of general type [46, Section 8], varieties of dimension ≤ 3 rationally dominated by products of curves [44, Example 3.15], varieties X with Abel–Jacobi trivial Chow groups (i.e. $A_{AJ}^i(X) = 0$ for all i) [43, Theorem 4], products of varieties with finite-dimensional motive [28].

3 Surfaces that are ρ -maximal

Definition 4 ([3]) A smooth projective variety X is said to be ρ -maximal if the rank ρ of the Neron–Severi group is equal to the Hodge number $h^{1,1}$.

Proposition 5 *Let X be a smooth projective variety of dimension 2 with $p_g = 1$ and $q = 0$. Assume that X has finite-dimensional motive, and that X is ρ -maximal. Then for any $z, z' \in A_{hom}^2(X)$, one has*

$$z \times z' = z' \times z \text{ in } A^4(X \times X) .$$

Proof Let ι denote the involution on $X \times X$ exchanging the two factors. The action of ι on cohomology is well-understood:

Lemma 6 *Let X be a surface with $q = 0$ and $p_g = 1$. We have*

$$(\Delta_{X \times X} - \Gamma_\iota)_* H^4(X \times X) \subset H^4(X \times X) \cap F^1$$

(here F^* denotes the Hodge filtration on $H^*(-, \mathbb{C})$).

Proof The only summand in the Künneth decomposition of $H^4(X \times X)$ that is not in F^1 is $H^2 X \otimes H^2 X$. The correspondence

$$(\Delta_{X \times X} - \Gamma_\iota)$$

acts on

$$\text{Im}(H^2 X \otimes H^2 X \rightarrow H^4(X \times X))$$

as twice the projector onto $\wedge^2 H^2 X$. The lemma now follows from the following, which is [53, Lemma 4.36].

Lemma 7 *Let H be a Hodge structure of weight n and with $\dim H^{n,0} = 1$. Then the Hodge structure of weight $2n$ on $\wedge^2 H$ has coniveau ≥ 1 .*

The ρ -maximality condition is used in the following guise:

Proposition 8 *Let X be a ρ -maximal surface. Let α be a Hodge class*

$$\alpha \in \left((H^4(X \times X) \cap F^1) \otimes (H^4(X \times X) \cap F^1) \right) \cap F^4 .$$

Then there exists a divisor $D \subset X \times X$, and a cycle class $\gamma \in A_4(D \times D)$ such that

$$\gamma = \alpha \text{ in } H^8(X^4) .$$

Proof Let

$$h^2 = h_{alg}^2 \oplus t_2(X)$$

denote the decomposition of Chow motives as in [27], i.e. $t_2(X) = (X, \pi_2^{tr}, 0)$ is the transcendental motive of X in the sense of loc. cit. Then the second cohomology group decomposes

$$H^2(X) = NS(X) \oplus W ,$$

where $W = H^2(t_2(X))$. The ρ -maximality of X implies that W is a 2-dimensional \mathbb{Q} -vector space, since

$$W_{\mathbb{C}} = H^{0,2}(X) \oplus H^{2,0}(X) .$$

We have that

$$\begin{aligned} H^4(X \times X) \cap F^1 &= (H^2(X) \otimes H^2(X)) \cap F^1 \\ &= NS(X) \otimes NS(X) \oplus NS(X) \otimes W \oplus W \otimes NS(X) \oplus (W \otimes W) \cap F^1 \\ &= NS(X) \otimes NS(X) \oplus NS(X) \otimes W \oplus W \otimes NS(X) \oplus (W \otimes W) \cap F^2 . \end{aligned}$$

It is easy to prove the Hodge conjecture for $(W \otimes W) \cap F^2$:

Lemma 9 *The \mathbb{Q} -vector space*

$$(W \otimes W) \cap F^2 \subset H^4(X \times X) \cap F^2$$

is of dimension 1, and generated by the cycle $\pi_2^{tr} \in A^2(X \times X)$.

Proof The complex vector space

$$F^2(W_{\mathbb{C}} \otimes W_{\mathbb{C}}) = H^{0,2}(X) \otimes H^{2,0}(X) \oplus H^{2,0}(X) \otimes H^{0,2}(X)$$

is 2-dimensional, with generators c, d such that $c = \bar{d}$. Let

$$a \in (W \otimes W) \cap F^2 ,$$

i.e. a is such that the complexification $a_{\mathbb{C}} \in H^4(X \times X, \mathbb{C})$ can be written

$$a_{\mathbb{C}} = \lambda c + \mu \bar{c} .$$

But the class $a_{\mathbb{C}}$, coming from rational cohomology, is invariant under conjugation, so that $\lambda = \mu$, i.e.

$$\dim(W \otimes W) \cap F^2 = 1 .$$

The class of the cycle π_2^{tr} in $H^4(X \times X)$ lies in $W \otimes W$ because $W = H^2(t_2(X)) = (\pi_2^{tr})_* H^2(X)$.

By assumption, α is a Hodge class in

$$\begin{aligned} & (H^4(X \times X) \cap F^1) \otimes (H^4(X \times X) \cap F^1) \\ &= \left(NS(X) \otimes NS(X) \oplus \cdots \oplus (W \otimes W) \cap F^2 \right) \otimes \left(NS(X) \otimes NS(X) \oplus \cdots \oplus (W \otimes W) \cap F^2 \right). \end{aligned}$$

It follows that α decomposes as a sum of Hodge classes $\alpha_1 + \cdots + \alpha_{16}$ in the various components; we now analyze the various components that occur.

First, suppose there is a factor $NS(X)$ both in the first half and in the second half of the decomposition, e.g. consider

$$\alpha_6 \in NS(X) \otimes W \otimes NS(X) \otimes W.$$

This class α_6 can be written

$$\alpha_6 = D_1 \times D_2 \times \hat{\alpha}_6 \in H^8(X^4),$$

with $D_1, D_2 \in NS(X)$ and $\hat{\alpha}_6 \in W \otimes W$. Since α_6 is a Hodge class, so is $\hat{\alpha}_6$. But then $\hat{\alpha}_6$ is algebraic, by lemma 9. It thus follows that α_6 is represented by a cycle supported on divisor times divisor in X^4 .

Next, suppose there is a factor $NS(X)$ on one side but not on the other side, e.g. consider

$$\alpha_8 \in NS(X) \otimes W \otimes (W \otimes W) \cap F^2.$$

Then the class α_8 can be written as

$$\alpha_8 = D \times \hat{\alpha}_8 \times t(X) \in H^8(X^4).$$

Now $\hat{\alpha}_8$ is a Hodge class in W , so it must be 0. The remaining cases are treated similarly.

Proposition 5 is now easily proven: Let $\pi_2 \in A^2(X \times X)$ denote a Chow–Künneth projector [33], [27]. Using lemma 6 and proposition 8, one obtains an equality between algebraic cycles modulo homological equivalence:

$$(\Delta_{X \times X} - \Gamma_\iota) \circ (\pi_2 \times \pi_2) = \gamma \text{ in } H^8(X^4),$$

where γ is a cycle supported on $D \times D$, for some divisor $D \subset X \times X$. This is equivalent to

$$(\pi_2 \times \pi_2) - \Gamma_\iota \circ (\pi_2 \times \pi_2) - \gamma = 0 \text{ in } H^8(X^4).$$

Using the nilpotence theorem (theorem 2), this implies there exists $N \in \mathbb{N}$ such that

$$\left((\pi_2 \times \pi_2) - \Gamma_\iota \circ (\pi_2 \times \pi_2) - \gamma \right)^{\circ N} = 0 \text{ in } A^4(X^4). \quad (1)$$

Without loss of generality, we may suppose N is odd. Define an integer

$$M := 1 + \binom{N}{2} + \binom{N}{4} + \cdots + \binom{N}{N-1} = 1 + \binom{N}{N-2} + \binom{N}{N-4} + \cdots + \binom{N}{1}.$$

Upon developing (1), we find an equality of correspondences

$$M\pi_2 \times \pi_2 - M\Gamma_\iota \circ (\pi_2 \times \pi_2) = \sum_{\ell} Q_{\ell} \text{ in } A^4(X^4), \quad (2)$$

where each $Q_{\ell} \in A^4(X^4)$ is a finite composition of correspondences

$$Q_{\ell} = Q_{\ell}^1 \circ \cdots \circ Q_{\ell}^{N'} \in A^4(X^4)$$

for $N' \leq N$, where $Q_{\ell}^j \in \{(\pi_2 \times \pi_2), \Gamma_\iota \circ (\pi_2 \times \pi_2), \gamma\}$, and at least one Q_{ℓ}^j is equal to γ . The correspondence γ (being supported on $D \times D$ for some divisor D) does not act on 0-cycles, so that

$$(Q_{\ell})_* A^4(X \times X) = 0 \text{ for all } Q_{\ell}.$$

Applying equation (2) to 0-cycles, we thus find that

$$(M(\pi_2 \times \pi_2 - \Gamma_\ell \circ (\pi_2 \times \pi_2)))_* A^4(X \times X) = 0 ,$$

i.e.

$$(\pi_2 \times \pi_2)_* = (\Gamma_\ell \circ (\pi_2 \times \pi_2))_* : A^4(X \times X) \rightarrow A^4(X \times X) .$$

Since $\pi_2 \times \pi_2$ acts as the identity on cycles of type $z \times z'$ with $z, z' \in A_{hom}^2 X$, we have thus proven that

$$z \times z' = z' \times z \text{ in } A^4(X \times X) ,$$

i.e. conjecture 1 is true for X .

Remark 10 In particular, it follows from proposition 5 that a $K3$ surface with Picard number 20 verifies conjecture 1; we will prove a more general result later (corollary 22). For surfaces of general type with $p_g = K_X^2 = 1$, Beauville shows [3, Proposition 9] that the ρ -maximal surfaces are dense in the moduli space. It would be interesting to prove that these surfaces have finite-dimensional motive.

Remark 11 In [7], Bonfanti constructs 2 families of surfaces of general type to which proposition 5 applies. These are the surfaces of type b and of type d in [7, Table 1], studied in detail in [7, Sections 3.1 and 3.3]. All surfaces studied in [7] are dominated by products of curves and, as such, they have finite-dimensional motive. The ρ -maximality of the surfaces of type b and of type d is established in [7, Section 4.1].

4 Some special $K3$ surfaces

4.1 Double planes

Proposition 12 (Voisin [47]) *Let X be a desingularization of the double cover of \mathbb{P}^2 branched along the union of two irreducible cubics. Then conjecture 1 is true for X .*

Proof This is [47, Theorem 3.4] (cf. also [53, Section 4.3.5.2], [51, Section 3]). Because we will use essentially the same argument in proposition 14 below, we briefly review Voisin's proof. Let

$$f_1(x), f_2(x)$$

denote the equations of the two plane cubics, where $x = [x_0 : x_1 : x_2] \in \mathbb{P}^2$. Let Σ be the surface defined by

$$\Sigma = \{[u : x_0 : x_1 : x_2] \in \mathbb{P}^3 \mid u^6 = f_1(x)f_2(x)\} \subset \mathbb{P}^3 .$$

There is a degree 3 covering

$$\begin{aligned} \psi : \Sigma &\rightarrow X , \\ (u, x) &\mapsto (u^3, x) \end{aligned}$$

(this corresponds to the quotient map $\mathbb{P}^3 \rightarrow \mathbb{P}(1, 1, 1, 3)$, since X can be seen as the hypersurface in weighted projective space $\mathbb{P}(1, 1, 1, 3)$ given by $v^2 = f_1(x)f_2(x)$). Let $W \subset \mathbb{P}^5$ be the sextic fourfold defined by

$$f_1(x)f_2(x) - f_1(y)f_2(y) = 0 ,$$

where $[x_0 : x_1 : x_2 : y_0 : y_1 : y_2]$ are homogeneous coordinates for \mathbb{P}^5 . Let $\widetilde{W} \rightarrow W$ denote a desingularization. The fourfold W is obviously invariant under the natural involution

$$\begin{aligned} i : \mathbb{P}^5 &\rightarrow \mathbb{P}^5 , \\ [x : y] &\mapsto [y : x] ; \end{aligned}$$

likewise, \widetilde{W} is \widetilde{i} -invariant, where \widetilde{i} is the induced involution.

There exists a (Shioda–style [40]) rational map

$$\begin{aligned} \phi: \Sigma \times \Sigma &\dashrightarrow W, \\ ([u : x], [u' : x']) &\mapsto [u'x : ux']; \end{aligned}$$

resolving indeterminacies one obtains a morphism

$$\tilde{\phi}: \widetilde{\Sigma \times \Sigma} \rightarrow \widetilde{W}.$$

We now have defined morphisms

$$\begin{array}{ccc} \widetilde{\Sigma \times \Sigma} & & \xrightarrow{\tilde{\phi}} \widetilde{W} \\ \tilde{\psi} \times \tilde{\psi} \downarrow & & \\ X \times X & & \end{array}$$

This induces a correspondence

$$\Gamma \in A^4(X \times X \times \widetilde{W}),$$

with action

$$\Gamma_* = \tilde{\phi}_*(\tilde{\psi} \times \tilde{\psi})^*: A_i(X \times X) \rightarrow A_i(\widetilde{W}).$$

Analyzing the action of Γ , one directly checks that

$$\Gamma_*: (A_0^{\text{hom}}(X) \otimes A_0^{\text{hom}}(X)) \rightarrow A_0(\widetilde{W})$$

is injective, and that

$$\Gamma_*(a \times a' - a' \times a) \subset A_0(\widetilde{W})^-,$$

for any $a, a' \in A_0^{\text{hom}}(X)$, where $A_0(\widetilde{W})^-$ denotes the -1 -eigenspace for the action of \tilde{i} [47, Lemma 3.4.1] (cf. also [51, Lemma 3.5] for a slight variant, where a different involution on \widetilde{W} is used).

It remains to prove that the eigenspace $A_0(\widetilde{W})^-$ is 0. To see this, one remarks that W is covered by the family of (Calabi–Yau) 3-folds W_α , where for each $\alpha \in \mathbb{C}$, one defines

$$W_\alpha := \{[x : y] \in \mathbb{P}^5 \mid f_1(x) = \alpha f_2(y), f_1(y) = \alpha f_2(x)\}.$$

Each W_α is i -invariant, and the general W_α is smooth. As each 0-cycle on W can be supported on finitely many smooth W_α 's, the vanishing of the eigenspace $A_0(\widetilde{W})^-$ follows from the following result:

Proposition 13 *Let $Z \subset \mathbb{P}^5$ be a 3-fold defined by two i -invariant cubic equations. Then $A_0(Z)^- = 0$.*

Proof This can be proven “by hand” using the method of [48].

Proposition 14 *Let X be a desingularization of the double cover of \mathbb{P}^2 branched along the union of an irreducible quartic and an irreducible quadric. Then conjecture 1 holds for X .*

Proof This is similar to the above. Let

$$f_1(x), f_2(x)$$

be equations for the quartic resp. quadric in the branch locus, where $x = [x_0 : x_1 : x_2]$. Let W be the fourfold defined by

$$f_1(x)f_2(x) - f_1(y)f_2(y) = 0.$$

As f_1f_2 is of even degree, W is invariant under the involution

$$\begin{aligned} \tau: W &\rightarrow W, \\ [x : y] &\mapsto [x : -y]. \end{aligned}$$

We let $\widetilde{W} \rightarrow W$ denote a resolution of singularities, and $\tilde{\tau}$ the induced involution. As above, there is a correspondence

$$\Gamma \in A^4(X \times X \times \widetilde{W}),$$

inducing an injection

$$\Gamma_* : \left(A_0^{\text{hom}}(X) \otimes A_0^{\text{hom}}(X) \right) \rightarrow A_0(\widetilde{W}) .$$

We proceed to check that

$$\Gamma_* \left(a \times a' - a' \times a \right) \subset A_0(\widetilde{W})^- ,$$

for any $a, a' \in A_0^{\text{hom}}(X)$, where now $A_0(\widetilde{W})^-$ denotes the -1 -eigenspace for the action of $\widetilde{\tau}$. To see this, note that Voisin [51, Lemma 3.5] proves that

$$\Gamma_* \left(a \times a' - a' \times a \right) \subset A_0(\widetilde{W})$$

is invariant under the involution \widetilde{j} induced by

$$\begin{aligned} j : W &\rightarrow W , \\ [x : y] &\mapsto [y : -x] \end{aligned}$$

(this involution j is denoted i in loc. cit.). Note that we also have, as above in the proof of proposition 12, that

$$\Gamma_* \left(a \times a' - a' \times a \right) \subset A_0(\widetilde{W})$$

is anti-invariant under the involution i exchanging x and y . Since

$$\tau = i \circ j ,$$

it follows that

$$\Gamma_* \left(a \times a' - a' \times a \right) \subset A_0(\widetilde{W})$$

is anti-invariant under $\widetilde{\tau}$, as claimed.

It only remains to prove that $A_0(\widetilde{W})^-$, the anti-invariant part under $\widetilde{\tau}$, vanishes. To this end, we consider a family of (Calabi–Yau) 3-folds W_α covering W , defined as

$$W_\alpha := \{ [x : y] \in \mathbb{P}^5 \mid f_1(x) = \alpha f_1(y), f_2(y) = \alpha f_2(x) \} .$$

Each W_α is τ -invariant (since f_1, f_2 are of even degree), and the general W_α is smooth. As each 0-cycle on W can be supported on finitely many smooth W_α ’s, the vanishing of the eigenspace $A_0(\widetilde{W})^-$ now follows from the following result:

Proposition 15 *Let $Z \subset \mathbb{P}^5$ be a smooth 3-fold defined by two τ -invariant equations of degree 2 and 4. Then $A_0(Z)^- = 0$.*

Proof Note that Z is Calabi–Yau, and the involution τ acts as the identity on $H^{3,0}(Z)$, i.e.

$$H^3(Z)^- \subset F^1 H^3(Z) .$$

One invokes [48, Proposition 2.1] to conclude that one has moreover

$$H^3(Z)^- \subset N^1 H^3(Z) ;$$

what’s more, $H^3(Z)^-$ is “parametrized by algebraic cycles” in the sense of [51]. Now one can apply the “spreading out” method of Voisin’s [50], [51] to the family of all smooth τ -invariant complete intersections of multi-degree $(2, 4)$. Some care is needed because one does not have a complete linear system; this problem can be overcome as in [51, Theorem 3.3].

Alternatively, one could prove proposition 15 “by hand” along the lines of [48].

Proposition 16 *Let X be a desingularization of the double cover of \mathbb{P}^2 branched along 6 lines in general position. Then conjecture 1 is true for X .*

Proof While this can probably be proven “directly” in the spirit of Voisin’s result (proposition 12), we prefer to give a somewhat more “fancy” proof. This proof hinges on the fact that the Kuga–Satake construction for X is algebraic [35]. More precisely, according to Paranjape [35] there exist an abelian variety A of dimension g and a correspondence $\Gamma' \in A^2(X \times A \times A)$ such that

$$(\Gamma')_*: T_X \rightarrow H^2(A \times A)$$

is an injection. It follows that there is an injection

$$\Gamma': t_2(X) \rightarrow h^2(A \times A) \text{ in } \mathcal{M}_{num},$$

where $t_2(X)$ is the transcendental motive of X in the sense of [27], and \mathcal{M}_{num} is the category of motives modulo numerical equivalence. Composing with some Lefschetz operator, one also gets an injection

$$\Gamma: t_2(X) \rightarrow h^{4g-2}(A \times A) \text{ in } \mathcal{M}_{num}$$

(here Γ is the composition $L^{2g-2} \circ \Gamma'$, where L is an ample line bundle on $A \times A$).

The category \mathcal{M}_{num} being semi-simple [24], this is a split injection, i.e. there exists a correspondence $\Psi \in A^2(A \times A \times X)$ such that

$$\Psi \circ \Gamma = \text{id}: t_2(X) \rightarrow t_2(X) \text{ in } \mathcal{M}_{num}.$$

But the motive $t_2(X)$ is finite-dimensional (it is a direct summand of $h(X)$, which is finite-dimensional since X is dominated by a product of curves [35]). This implies that there exists $N \in \mathbb{N}$ such that

$$(\Delta - \Psi \circ \Gamma)^{\circ N} = 0: t_2(X) \rightarrow t_2(X) \text{ in } \mathcal{M}_{rat},$$

and hence that

$$\Gamma_*: A_{hom}^2(X) = A_{AJ}^2(X) = A^2(t_2(X)) \rightarrow A_{AJ}^{2g}(A \times A)$$

is injective. We note that, by construction, the action of Γ on Chow groups factors as

$$\Gamma_*: A_{AJ}^2(X) \xrightarrow{\Gamma'} A^2(A \times A) \xrightarrow{L^{2g-2}} A^{2g}(A \times A).$$

Let $A_{(*)}^*(\cdot)$ denote Beauville’s filtration on Chow groups of abelian varieties [2]. It follows that

$$\Gamma_*(A_{AJ}^2(X)) \subset \bigoplus_{j \leq 2} A_{(j)}^{2g}(A \times A),$$

as the Lefschetz operator preserves Beauville’s filtration [30]. On the other hand,

$$\Gamma_*(A_{AJ}^2(X)) \subset A_{AJ}^{2g}(A \times A) = \bigoplus_{j \geq 2} A_{(j)}^{2g}(A \times A).$$

The conclusion is that there is an injection

$$\Gamma_*: A_{AJ}^2(X) \rightarrow A_{(2)}^{2g}(A \times A).$$

The same argument gives also that

$$\Gamma \times \Gamma: \text{Im}(A_{hom}^2(X) \otimes A_{hom}^2(X) \rightarrow A^4(X \times X)) \subset A^4(t_2(X) \otimes t_2(X)) \rightarrow A^{4g}(A^4)$$

is injective. It now suffices to prove a statement for the abelian variety $B = A \times A$:

Proposition 17 *Let B be an abelian variety of dimension $2g$. Let*

$$a, a' \in A_{(2)}^{2g}(B)$$

be 2 0-cycles. Then

$$a \times a' - a' \times a = 0 \text{ in } A^{4g}(B \times B).$$

Proof The group $A_{(2)}^{2g}(B)$ is generated by products of divisors

$$D_1 \cdot D_2 \cdot \dots \cdot D_{2g} \in A^{2g}(B),$$

with 2 of the D_j in $A_{(1)}^1(B) = \text{Pic}^0(B)$, and the remaining $2g - 2$ D_j in $A_{(0)}^1(B)$ [4]. As in [53, Example 4.40], we consider the map

$$\sigma: B \times B \rightarrow B \times B, (a, b) \mapsto (a + b, a - b).$$

This is an isogeny, and one can check it induces a homothety on $A^*(B \times B)$. But on the other hand,

$$\sigma \circ \iota \circ \sigma = 2(\text{id}_B, -\text{id}_B): B \times B \rightarrow B \times B.$$

It thus suffices to note that

$$(\text{id}_B, -\text{id}_B)_*(D_1 \cdot \dots \cdot D_{2g} \times D'_1 \cdot \dots \cdot D'_{2g}) = D_1 \cdot \dots \cdot D_{2g} \times D'_1 \cdot \dots \cdot D'_{2g} \text{ in } A^{4g}(B \times B),$$

since there is an even number of divisors D'_j for which $(-\text{id}_B)_*(D'_j) = -D'_j$ in $A^1 B$.

Remark 18 Note that the proof of proposition 16 actually establishes something more general: if X is a $K3$ surface with finite-dimensional motive, and the Kuga–Satake embedding of X is induced by an algebraic cycle, then conjecture 1 is true for X . For instance, this also applies to the quartic surface X in \mathbb{P}^3 defined by an equation

$$t^4 = f(x, y, z),$$

where it is supposed that $f(x, y, z) = 0$ defines a smooth quartic curve in \mathbb{P}^2 . (Indeed, the construction in [17, Example 11.3] (where this example is attributed to Nori) shows that both hypotheses are fulfilled by X : the “Kuga–Satake Hodge conjecture” is shown to hold, and it is shown that X is dominated by a product of curves so the motive is finite-dimensional.) Another example satisfying these conditions is [18, Example 3.11], which is a 9-dimensional family of elliptic $K3$ surfaces.

Remark 19 Improving on the results of this subsection, it would be interesting to consider more generally $K3$ surfaces that are double covers of \mathbb{P}^2 ramified along an irreducible sextic. Voisin [51] proposes a tentative strategy towards settling conjecture 1 for these $K3$ surfaces: applying [51, Lemma 3.5] combined with (an improved variant of) [51, Theorem 0.6], it would suffice to prove that for a certain sextic fourfold Y associated to X , one has that $F^1 H^4(Y)$ is “parametrized by algebraic cycles of dimension 1”, in the sense of [51] (that is, it would suffice to prove a strong form of the generalized Hodge conjecture for Y).

4.2 Shioda–Inose structure

Definition 20 ([32]) For any surface M , let $T_M \subset H^2(M, \mathbb{Z})$ denote the transcendental lattice. For $\ell \in \mathbb{N}$, let $T_M(\ell)$ denote the lattice T_M with intersection form multiplied by ℓ . A *Nikulin involution* on a $K3$ surface X is an involution acting as the identity on $H^{0,2}(X)$.

A $K3$ surface X admits a *Shioda–Inose structure* if there exists a Nikulin involution i on X with rational quotient map

$$\pi: X \dashrightarrow Y$$

where Y is a Kummer surface, and π_* induces a Hodge isometry $T_X(2) \cong T_Y$.

Proposition 21 *Let X be a $K3$ surface with a Shioda–Inose structure. Then conjecture 1 is true for X .*

Proof As the Nikulin involution i acts as the identity on $A^2 X$ [49], there is an isomorphism

$$\pi^*: A_{\text{hom}}^2(Y) \xrightarrow{\cong} A_{\text{hom}}^2(X).$$

The result now follows from the truth of conjecture 1 for the Kummer surface Y [47].

Corollary 22 *Let X be a $K3$ surface with Picard number ≥ 19 . Then conjecture 1 is true for X .*

Proof X has a Shioda–Inose structure [32, Corollary 6.4].

Remark 23 $K3$ surfaces admitting a Shioda–Inose structure are very special: their Picard number is at least 17. For the case of Picard number 17, explicit families of $K3$ surfaces with Shioda–Inose structure have been discovered: these are certain elliptic fibrations [29], [19, 4.7], as well as double covers of the plane branched along certain singular sextics [19, 4.5]. More elliptic fibrations with a Shioda–Inose structure are given by [10, Theorem 4.4].

Note that a $K3$ surface admitting a Shioda–Inose structure and with Picard number 17 or 18 can not be a Kummer surface [16, Corollary 3.7].

Remark 24 It seems interesting to study conjecture 1 in positive characteristic as well. As a starter, we note that corollary 22 still holds in positive characteristic, thanks to work of Liedtke [31]. More precisely, let X be a $K3$ surface over an algebraically closed field of characteristic ≥ 5 . If the Picard number of X is 22, X is unirational [31, Theorem 5.3] so $A^2(X)$ is trivial. The Picard number can not be 21 [31, Theorem 2.6]. If the Picard number is 19 or 20, X is dominated by a Kummer surface [31, Theorem 2.6], and the result follows since the result on abelian varieties [53, Example 4.40] still hold in positive characteristic.

4.3 Nikulin involutions

There are many $K3$ surfaces X with a Nikulin involution i that is *not* a Shioda–Inose structure (e.g., when the quotient $K3$ surface is not a Kummer surface). Sometimes, we are lucky and the quotient $K3$ surface (more precisely, a minimal resolution of X/i) is one for which conjecture 1 is known. In these cases, it follows that conjecture 1 also holds for X . We give 2 examples of this phenomenon; one is a family of $K3$ s with Picard number 9, the other family has Picard number 16.

Proposition 25 *Let X be a $K3$ surface such that the Neron–Severi group is isomorphic to the lattice $\Lambda_{\tilde{4}}$, in the notation of [19]. Then conjecture 1 is true for X .*

Proof The 11-dimensional family $\mathcal{M}_{\tilde{4}}$ of $K3$ surfaces of this type is described explicitly in [19, 3.5]. In particular, it is shown in loc. cit. that there exists a Nikulin involution i on X such that a minimal resolution of the quotient X/i is a $K3$ surface Y isomorphic to a double plane with branch locus the union of a quartic and a conic. Conjecture 1 is verified for such Y (proposition 12). Since pull-back induces an isomorphism $A_{hom}^2(Y) \cong A_{hom}^2(X)$ [49], it follows that conjecture 1 holds for X .

Proposition 26 *Let X be a generic $K3$ surface polarized by the lattice $H \oplus E_7 \oplus E_7$, in the sense of [10]. Then conjecture 1 is true for X .*

Proof According to [10, Theorem 4.4], there is a Nikulin involution i on X such that a minimal resolution of the quotient X/i is a $K3$ surface Y isomorphic to a double cover of the plane branched along 6 lines. Conjecture 1 holds for Y (proposition 16). Since pull-back induces an isomorphism $A_{hom}^2(Y) \cong A_{hom}^2(X)$ [49], it follows that conjecture 1 holds for X .

5 Kunev surfaces

In this section we show that conjecture 1 is true for Kunev surfaces. These surfaces form a 12-dimensional family of surfaces of general type with $p_g = K_X^2 = 1$. The proof is quite direct, and goes as follows. The bicanonical map of a Kunev surface factors over a $K3$ surface, which is of a special type: it is obtained from a double cover of \mathbb{P}^2 branched along the union of 2 smooth cubics [41]. By chance, for such $K3$ surfaces Voisin has already established the truth of conjecture 1 ([47] or proposition 12). Hence, to prove conjecture 1 for the Kunev surface X , it only remains to relate 0-cycles on X and 0-cycles on the associated $K3$ surface; this can be done using the “spreading out” argument of [50] and [51].

Definition 27 ([41]) A *Kunev surface* is a smooth projective surface X of general type with $p_g(X) = 1$, $K_X^2 = 1$, such that its unique effective canonical divisor is a smooth curve, and the morphism given by $|2K_X|$ is a Galois covering of \mathbb{P}^2 .

Remark 28 Surfaces of general type with $p_g = K_X^2 = 1$ are studied in [9] and [41]. In [9], a Kunev surface is called a *special* surface with $p_g = K_X^2 = 1$.

Proposition 29 *Let X be a Kunev surface. Then conjecture 1 is true for X .*

Proof According to the structural results of [41] (or, independently, [9]), any surface of general type with $p_g = K_X^2 = 1$ is a complete intersection of multidegree $(6, 6)$ in a weighted projective space $P := \mathbb{P}(1, 2, 2, 3, 3)$. If in addition X is a Kunev surface, then it is proven in [9] and [41] that the equations defining X are invariant under the involution

$$i: P \rightarrow P, \\ [x_0 : x_1 : \dots : x_4] \mapsto [-x_0 : x_1 : \dots : x_4].$$

The quotient $Y = X/i$ is a $K3$ surface, which is obtained by desingularizing a double cover of \mathbb{P}^2 branched along two smooth cubics. Conjecture 1 is true for Y [47, Theorem 3.4]. This implies conjecture 1 for X , provided we can relate 0-cycles on X to 0-cycles on Y ; this is done in proposition 30 below.

Proposition 30 *Let X be a Kunev surface, and let $p: X \rightarrow Y$ denote the quotient map to the associated $K3$ surface. Then*

$$p^*: A_{hom}^2(Y) \rightarrow A_{hom}^2(X)$$

is an isomorphism.

Proof We use the “spreading out” argument of Voisin’s [50], [51], which exploits the fact that the surfaces come in a family. Let

$$\pi: \mathcal{X} \rightarrow B$$

denote the family of all smooth complete intersections in $P := \mathbb{P}(1, 2, 2, 3, 3)$, defined by 2 equations of weighted degree 6 where x_0 only occurs in even degree. For any $b \in B$, let X_b denote the fibre $\pi^{-1}(b)$. The involution i induces an involution on the total space of the family, which we still denote by i . This induces a quotient map

$$p: \mathcal{X} \rightarrow \mathcal{Y} := \mathcal{X}/i,$$

where $\mathcal{Y} \rightarrow B$ is the family of associated $K3$ surfaces.

Consider now the cycle

$$\mathcal{D} := \Delta - \frac{1}{2} \Gamma_p \in A^2(\mathcal{X} \times_B \mathcal{X})$$

(where Δ denotes the relative diagonal, and Γ_p is the graph of p). This cycle has the property that for any $b \in B$, the restriction

$$\mathcal{D}|_{X_b \times X_b} \in H^4(X_b \times X_b)$$

is supported on $Z_b \times Z_b$, for some divisor $Z_b \subset X_b$. (Indeed, for any $b \in B$ we have that

$$(p_b)_*(p_b)^*(p_b)_* = 2(p_b)_*: H^{2,0}(X_b) \rightarrow H^{2,0}(Y_b),$$

and hence

$$(p_b)^*(p_b)_* = 2\text{id}: H^{2,0}(X_b) \rightarrow H^{2,0}(X_b).)$$

Using Voisin’s “spreading out” result [50, Proposition 2.7], it follows there exists a divisor $\mathcal{Z} \subset \mathcal{X}$ and a cycle $\mathcal{D}' \in A^2(\mathcal{X} \times_B \mathcal{X})$ supported on $\mathcal{Z} \times_B \mathcal{Z}$, such that

$$(\mathcal{D} - \mathcal{D}')|_{X_b \times X_b} = 0 \text{ in } H^4(X_b \times X_b),$$

for all $b \in B$. Next, an analysis of the Leray spectral sequence as in [50, Lemma 2.12] shows that there exists a cycle \mathcal{D}'' with support on $\mathcal{Z} \times_B \mathcal{X} \cup \mathcal{X} \times_B \mathcal{Z}$, such that we have the global homological vanishing

$$\mathcal{D}_{new} := \mathcal{D} - \mathcal{D}' - \mathcal{D}'' = 0 \text{ in } H^4(\mathcal{X} \times_B \mathcal{X})$$

(here we have enlarged the divisor $\mathcal{Z} \subset \mathcal{X}$). Denoting by f the blow-up of $\mathcal{X} \times_B \mathcal{X}$ along the relative diagonal, we also have

$$f^*(\mathcal{D}_{new}) = 0 \text{ in } H^4(\widetilde{\mathcal{X} \times_B \mathcal{X}}).$$

Let Q be the compactification of $\mathcal{X} \times_B \mathcal{X}$ introduced in lemma 31 below. The variety Q is almost smooth: it is a quotient variety $Q = Q'/G$, where G is a finite group (because P is a quotient variety). This implies there is a good intersection theory with rational coefficients on Q [14, Example 17.4.10]. Using the truth of the Hodge conjecture for divisors, we find there exists a cycle class

$$\overline{\mathcal{D}}_{new} \in A_{hom}^2(Q)$$

restricting to $f^*(\mathcal{D}_{new})$. But the cycle $\overline{\mathcal{D}}_{new}$ is rationally trivial (lemma 31), hence so is its restriction to any fibre. This proves proposition 30 for general $b \in B$: indeed, we find an equality

$$\Delta_{X_b} - \frac{1}{2} {}^t \Gamma_P \circ \Gamma_P = (\mathcal{D}' + \mathcal{D}'')|_{X_b \times X_b} \text{ in } A^2(X_b \times X_b),$$

and for general $b \in B$ the right-hand side does not act on $A_{hom}^2(X_b) = A_{AJ}^2(X_b)$.

To get the result for any $b_0 \in B$, it suffices to note that in the above construction, the divisor \mathcal{Z} supporting the cycles \mathcal{D}' and \mathcal{D}'' may be chosen in general position with respect to X_{b_0} , and then the above argument applies to X_{b_0} .

Lemma 31 *Set-up as above. Let*

$$f: \widetilde{\mathcal{X} \times_B \mathcal{X}} \rightarrow \mathcal{X} \times_B \mathcal{X}$$

be the blow-up along the relative diagonal, and let

$$\widetilde{P \times P} \rightarrow P \times P$$

be the blow-up along the diagonal. There exists a projective compactification

$$Q \supset \widetilde{\mathcal{X} \times_B \mathcal{X}},$$

with the property that Q is a fibre bundle over $\widetilde{P \times P}$, and fibres are products of projective spaces. In particular, we have

$$A_{hom}^2(Q) = 0.$$

Proof (This is inspired by Voisin's [50, proof of proposition 2.13] (cf. also [51, Lemma 1.3], [53, Lemma 4.32]), which treats the slightly different case of the complete family of smooth complete intersections defined by very ample line bundles in an ambient space with trivial Chow groups.)

A point of $\widetilde{P \times P}$ is a triple (x, y, z) , where $x, y \in P$ and z is a length 2 subscheme of $P \times P$ with $z = x + y$. Let $\bar{B} \supset B$ denote the product of projective spaces parametrizing all pairs of (not necessarily smooth) weighted homogeneous polynomials of degree 6 containing x_0 in even degree. The quasi-projective variety $\widetilde{\mathcal{X} \times_B \mathcal{X}}$ is contained in the projective variety $Q \subset \bar{B} \times \widetilde{P \times P}$ defined as

$$Q = \left\{ ((\sigma_1, \sigma_2), x, y, z) \in \bar{B} \times \widetilde{P \times P} \mid \sigma_1|_z = \sigma_2|_z = 0 \right\} \subset \bar{B} \times \widetilde{P \times P}.$$

Let $p: Q \rightarrow \widetilde{P \times P}$ denote the projection. The fibre of p over $(x, y, z) \in \widetilde{P \times P}$ is

$$p^{-1}(x, y, z) = \left\{ (\sigma_1, \sigma_2) \in \bar{B} \mid \sigma_1|_z = \sigma_2|_z = 0 \right\}.$$

We want to show that any fibre is a product of 2 codimension 2 linear subspaces in \bar{B} , i.e. that any z imposes 2 independent conditions on the polynomials σ_j . To this end, we note that there exists a degree 2 map

$$\phi: P = \mathbb{P}(1, 2, 2, 3, 3) \rightarrow \mathbb{P}(2, 2, 2, 3, 3) =: P',$$

and that the polynomials in \bar{B} correspond to

$$\bar{B}' := \phi^*|\mathcal{O}_{P'}(6)| \times \phi^*|\mathcal{O}_{P'}(6)|.$$

It follows that the fibre $p^{-1}(x, y, z)$ is isomorphic to the subspace of \bar{B}' of polynomials passing through $\phi(z)$. But $\mathcal{O}_{P'}(6)$ is a very ample line bundle on P' (this is proven in lemma 32 below), so this subspace has codimension 2.

The conclusion about the vanishing of $A_{hom}^2(Q)$ follows from the fact that blow-ups and fibre bundle structures preserve the property of having trivial Chow groups [50].

Lemma 32 *Let P' be the weighted projective space $\mathbb{P}(2, 2, 2, 3, 3)$. Then the line bundle $\mathcal{O}_{P'}(6)$ is very ample.*

Proof The coherent sheaf $\mathcal{O}_{P'}(6)$ is locally free, because 6 is a multiple of the “weights” 2 and 3 [13]. To see that this line bundle is very ample, we use the following numerical criterion:

Proposition 33 (Delorme [12]) *Let $P = \mathbb{P}(q_0, q_1, \dots, q_n)$ be a weighted projective space. Let m be the least common multiple of the q_j . Suppose every monomial*

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n}$$

of (weighted) degree km ($k \in \mathbb{N}^$) is divisible by a monomial of (weighted) degree m . Then $\mathcal{O}_P(m)$ is very ample.*

(This is the case $E(x) = 0$ of [12, Proposition 2.3(iii)].)

We apply proposition 33 to the set-up of lemma 32. A monomial of degree $6k$ is of the form $x^{\underline{b}} = x_0^{b_0} \dots x_4^{b_4}$ with

$$2(b_0 + b_1 + b_2) + 3(b_3 + b_4) = 6k.$$

Suppose $b_3 + b_4 \geq 2$. Then the condition is obviously fulfilled, since we have a degree 6 monomial x_3x_4 (or x_3^2 or x_4^2) dividing $x^{\underline{b}}$. So we may suppose $b_4 = 0$ and hence also $b_3 = 0$ (since $b_3 = 1$ would imply $6k$ is odd). Again, it is easily seen that the condition of the proposition is fulfilled: one can take an appropriate combination of x_0, x_1, x_2 to create a degree 6 monomial dividing $x^{\underline{b}}$.

Remark 34 There are two possible generalizations of proposition 29 that seem natural:

The first is to try and extend proposition 29 to all surfaces of general type with $p_g = K_X^2 = 1$. Such surfaces are complete intersections in a weighted projective space [41], [9], so Voisin’s method of spreading out cycles [50], [51] applies. The “only” two obstacles that need to be circumvented are (1) that one needs the generalized Hodge conjecture for the Hodge structure $\wedge^2 H^2(X) \subset H^4(X \times X)$, and (2) that one needs the Voisin standard conjecture [50, Conjecture 0.6] to get a cycle supported on some subvariety inside X^4 .

The other direction of generalization would be to extend proposition 29 to all *Todorov surfaces*, i.e. minimal surfaces X of general type with $q = 0$ and $p_g = 1$ having an involution i such that S/i is birational to a $K3$ surface and such that the bicanonical map of X is composed with i . A Kunev surface is a Todorov surface with $K_X^2 = 1$. For any Todorov surface X , one can prove [38] that the minimal resolution of X/i is a $K3$ surface Y obtained from a double plane with branch locus a union of 2 cubics. As conjecture 1 is known for such Y (proposition 12), it “only” remains to show that $A_{hom}^2(X) \cong A_{hom}^2(Y)$. For the Kunev surfaces of proposition 29, this was easy because they are complete intersections in a weighted projective space; for the other Todorov surfaces (i.e., with $K_X^2 > 1$), perhaps the total space of the family can likewise be exploited?

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